Thermal diffusion of Boussinesq solitons

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We consider the problem of the soliton dynamics in the presence of an external noisy force for the Boussinesq type equations. A set of ordinary differential equations (ODEs) of the relevant coordinates of the system is derived. We show that for the improved Boussinesq (IBq) equation the set of ODEs has limiting cases leading to a set of ODEs which can be directly derived either from the ill-posed Boussinesq equation or from the Korteweg-de Vries (KdV) equation. The case of a soliton propagating in the presence of damping and thermal noise is considered for the IBq equation. A good agreement between theory and simulations is observed showing the strong robustness of these excitations. The results obtained here generalize previous results obtained in the frame of the KdV equation for lattice solitons in the monatomic chain of atoms.

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I. INTRODUCTION

The Boussinesq (Bq) equation is a paradigm in the study of propagation of nonlinear pulses in a weakly dispersive medium. There are two basic dimensionless forms of this equation. Boussinesq [1] derived the first form,

$$\partial_t^2 y(x,t) = \partial_x^2 y(x,t) + \lambda \partial_x^4 y(x,t) + \partial_x(f(\partial_x y(x,t))), \qquad (1)$$

to describe shallow-water waves. In Eq. (1) $f(u(x,t))$ is a nonlinear force with $u(x,t)=\partial_x y(x,t)$, and λ is a dispersion parameter. Equation (1) is called the ill-posed Boussinesq (IPBq) equation [2,3] because its dispersion relation leads to a nonphysical instability of linear modes [2–4]. The IPBq equation arises in a wide variety of physical systems including propagation of waves in lattices modeling discrete microscopic structures [5,6]. Only some soliton-type solutions are known [7].

The IPBq can be approached by the so-called improved Bq (IBq) equation,

$$\partial_t^2 y(x,t) = \partial_x^2 y(x,t) + \lambda \partial_x^2 \partial_t^2 y(x,t) + \partial_x(f(\partial_x y(x,t))), \qquad (2)$$

which does not show instabilities for the linear modes [5,6,8].

Both Bq equations, Eqs. (1) and (2), admit supersonic one-soliton solutions which are nontopological [7,9]. We note that for low velocities, i.e., soliton velocities close to the sound velocity, the one-soliton solutions of the IBq equation can be approached by the solution of the IPBq equation. Besides, if one considers solitons traveling in some preferential direction the Bq equations can be approached by the one-soliton solution of the well-known Korteweg-de Vries (KdV) equation. In the regime of high velocities the one-soliton solution of the IBq equation is more accurate to describe discrete lattice solitons (solitons in monatomic chains) [6,9]. The Bq-type equations have been the subject of extensive investigations and some of the main features are rather well-established. However, the mathematical theory for such equations is not as complete as in the case of the KdV-type equations.

In this context we consider the problem of the soliton dynamics of forced Bq-type equations. There are a number of papers where forced Bq-type equations show up, for instance, in the context of the Cauchy problem [10], in the study of the effect of nonlocal interactions in anharmonic lattices [11,12], or in the study of two-dimensional lattices [13]. More recently, for instance, the soliton dynamics of the IBq equation in the presence of Stokes and hydrodynamical damping was studied numerically and analytically [9] by using the multiple-scale perturbation technique. In the present work we consider a simpler analytic technique, namely the collective coordinate approach [14–17], in order to study the soliton dynamics of the IBq equation in the presence of an external noisy force. We note that the collective coordinate approach has been very successful in the study of the dynamics of coherent excitations in the presence of external forces, for instance, stochastic vortex dynamics in the two-dimensional Heisenberg model [14], thermal diffusion of sine-Gordon solitons [15], soliton diffusion on the classical Heisenberg chain [16], or a chain of atoms under thermal fluctuations [17,18]. This technique takes advantage of the particle-nature of the coherent excitations, neglecting extended modes solution of the linearized stability problem. Here we must remark that to our knowledge the eigenvalue problem of the linearized stability equation has not been completely solved for Bq-type equations [19]. However, for cases where external forces are small enough those extended modes do not affect significantly the soliton dynamics [17,18].

In the present paper we study the forced Bq equation in the frame of the collective coordinate approach. Here we obtain ordinary differential equations (ODEs) of the relevant coordinates of the system. We show that the ODEs following from the IPBq equation as well as the ODEs following from the KdV equation are special cases of the IBq equation. We also discuss the effect of neglecting the extended modes in the case of the KdV equation. For information we show that the ODEs obtained from the collective coordinate approach can also be obtained by more elaborate methods as the adiabatic perturbation theory. Additionally we study as an ex-

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ample the thermal diffusion of the supersonic one-soliton solutions of the IBq equation. We show that ODEs in the frame of the collective coordinate approach can predict the behavior of the soliton position as well as its variance.

II. FORCED IBQ

The dimensionless forced IBq equation reads

$$\partial_t^2 y(x,t) = \partial_x^2 y(x,t) + \lambda \partial_x^2 \partial_t^2 y(x,t) + \partial_x(f(u(x,t))) + F, \quad (3)$$

where F is an external force, $y(x,t)$ is a field which can be associated, for instance, with the absolute displacement of a particle with respect to its equilibrium position in the monatomic chain. In the same way $u(x,t)=\partial_x y(x,t)$ can be associated with the relative displacement of nearest neighbors. In Eq. (3)

$$f(u) = \frac{dV(u)}{du} - u \quad (4)$$

and $V(u)$ is a potential, which in the case of the monatomic chain, can be associated with the interaction potential of nearest neighbors. In particular for the case of a harmonic potential with powerlike anharmonicity,

$$V(u) = \frac{1}{2}u^2 + \frac{1}{n}u^n \quad \text{with } n > 2, \quad (5)$$

the one-soliton solution of the homogeneous Eq. (2) has a kink shape, namely

$$y(x,t) = \begin{cases} \frac{3\lambda\eta}{1-\lambda\eta^2}\tanh\left(\frac{\eta}{2}(x-vt)\right), & n=3, \\ 2\sqrt{\frac{2\lambda}{1-\lambda\eta^2}}\arctan\left[\tanh\left(\frac{\eta}{2}(x-vt)\right)\right], & n=4, \end{cases} \quad (6)$$

where v is the soliton velocity and $\eta=\sqrt{(v^2-1)/(\lambda v^2)}$ is the inverse of the soliton width.

III. COLLECTIVE COORDINATE APPROACH

In order to achieve the calculations we reduce the second-order differential equation (3) to a set of two first-order differential equations, namely

$$\partial_t w(x,t) = \partial_x^2 y(x,t) + \lambda \partial_x^2 \partial_t^2 y(x,t) + \partial_x(f(\partial_x y(x,t))) + F,$$

$$\partial_t y(x,t) = w(x,t). \quad (7)$$

Notice that we do not write the dispersion term, $\partial_x^2\partial_t^2 y(x,t)$, in terms of w. Here we apply a traveling wave ansatz of the form

$$y(x,t) = y(x - X(t), \eta(t)),$$

$$w(x,t) = w(x - X(t), \eta(t)). \quad (8)$$

where $X(t)$ and $\eta(t)$ are the collective coordinates, which depend on time. In order to simplify the notation the explicit temporal dependence of the collective coordinates will not be written. Notice that $\partial_t = \dot{X}\partial_X + \dot{\eta}\partial_\eta$ and $\partial_x = -\partial_X$. By inserting Eqs. (8) into Eqs. (7) we get

$$\dot{X}\partial_X w + \dot{\eta}\partial_\eta w = \partial_X^2 y + \lambda \partial_X^2 (\dot{X}\partial_X + \dot{\eta}\partial_\eta)^2 y - \partial_X f(-\partial_X y) + F, \quad (9)$$

$$\dot{X}\partial_X y + \dot{\eta}\partial_\eta y = w, \quad (10)$$

where $\dot{} \equiv \frac{d}{dt}$. In order to obtain equations of motion for the coordinates X and η we follow Refs. [21,22]. So, in order to obtain an equation for $\dot{X}$ we multiply Eqs. (9) and (10) by $\partial_X y$ and $\partial_X w$, respectively, and then we subtract one from each other and integrate over x. In order to obtain the second equation of motion for $\dot{\eta}$ we proceed in a similar form, we multiply Eqs. (9) and (10) by $\partial_\eta y$ and $\partial_\eta w$, respectively, and finally we subtract one from each other and integrate, thus finding

$$A_i \dot{X} + B_i \dot{\eta} = T_i + T_i^{Ext}, \quad i = 1,2, \quad (11)$$

where $A_1 = B_2 = 0$, and

$$B_1 = \int dx(\partial_\eta y \partial_X w - \partial_\eta w \partial_X y), \quad (12)$$

$$T_1 = \int dx[-\lambda \partial_X y \partial_X^4 y \dot{X}^2 - 2\lambda \dot{\eta} \partial_X y \partial_\eta \partial_X^3 y \dot{X} + w \partial_X w - \partial_x f(x,t) \partial_X y - \partial_X y \partial_X^2 y - \lambda \ddot{\eta} \partial_X y \partial_\eta \partial_X^2 y - \lambda \dot{\eta}^2 \partial_X y \partial_\eta^2 \partial_X^2 y - \lambda \ddot{X} \partial_X y \partial_X^3 y], \quad (13)$$

$$T_1^{Ext} = -\int dx[F(x,t)\partial_X y], \quad (14)$$

$$A_2 = \int dx(\partial_\eta w \partial_X y - \partial_\eta y \partial_X w), \quad (15)$$

$$T_2 = \int dx[-\lambda \partial_\eta y \partial_X^4 y \dot{X}^2 - 2\lambda \dot{\eta} \partial_\eta y \partial_\eta \partial_X^3 y \dot{X} + w \partial_\eta w - \partial_\eta y \partial_x f(x,t) - \partial_\eta y \partial_X^2 y - \lambda \ddot{\eta} \partial_\eta y \partial_\eta \partial_X^2 y - \lambda \dot{\eta}^2 \partial_\eta y \partial_\eta^2 \partial_X^2 y - \lambda \ddot{X} \partial_\eta y \partial_X^3 y], \quad (16)$$

$$T_2^{Ext} = -\int dx[F(x,t)\partial_\eta y]. \quad (17)$$

From Eqs. (6) and its first time derivative we can write the ansatz for the cubic ($n=3$) and quartic ($n=4$) case, namely

$$y(x-X,\eta) = \begin{cases} \frac{3\lambda\eta}{1-\lambda\eta^2}\tanh\left(\frac{\eta}{2}(x-X)\right), & n=3, \\ 2\sqrt{\frac{2\lambda}{1-\lambda\eta^2}}\arctan\left[\tanh\left(\frac{\eta}{2}(x-X)\right)\right], & n=4. \end{cases}$$

$$w(x-X,\eta)=\begin{cases}-\dot{X}\dfrac{3\lambda\eta^2}{2(1-\lambda\eta^2)}\text{sech}^2\left(\dfrac{\eta}{2}(x-X)\right), & n=3,\\ -\dot{X}\sqrt{\dfrac{2\lambda\eta^2}{1-\lambda\eta^2}}\,\text{sech}[\eta(x-X)], & n=4.\end{cases} \tag{18}$$

For the cubic case ($n=3$) by inserting the ansatz (18) into Eqs. (11) we get

$$\dot{\eta}=-\frac{\lambda\eta^3(-1+\lambda\eta^2)}{(-15-10\lambda\eta^2+\lambda^2\eta^4)}\left(\frac{\ddot{X}}{\dot{X}}\right)+\frac{5(-1+\lambda\eta^2)^2\int_{-\infty}^{\infty}\text{sech}^2(z)F\,dz}{2\lambda\eta(-15-10\lambda\eta^2+\lambda^2\eta^4)\dot{X}}, \tag{19}$$

$$\begin{aligned}\dot{X}^2-1=&\frac{\lambda\eta^2}{1-\lambda\eta^2}-\frac{\lambda[30+\pi^2+(300+\pi^2)\lambda\eta^2-5(-30+\pi^2)\lambda^2\eta^4+3\pi^2\lambda^3\eta^6]}{3(-1+\lambda\eta^2)^2(-15-10\lambda\eta^2+\lambda^2\eta^4)}\left(\frac{\dot{\eta}}{\eta}\right)^2\\&+\frac{2\lambda[30+\pi^2-2(-15+\pi^2)\lambda\eta^2+\pi^2\lambda^2\eta^4]}{3(-1+\lambda\eta^2)(-15-10\lambda\eta^2+\lambda^2\eta^4)}\left(\frac{\ddot{\eta}}{\eta}\right)-\frac{10(-1+\lambda\eta^2)}{\lambda\eta^3(-15-10\lambda\eta^2+\lambda^2\eta^4)}\\&\times\int_{-\infty}^{\infty}\{z\,\text{sech}^2(z)+\tanh(z)+\lambda[-z\,\text{sech}^2(z)+\tanh(z)]\eta^2\}F\,dz,\end{aligned} \tag{20}$$

with $z=\eta(x-X)/2$.

Up to now we did not impose any condition on the external force F. However, the ODEs (19) and (20) are coupled in a rather complicated way; notice that Eqs. (19) and (20) have $\dot{X}$ and $\dot{\eta}$ on the left-hand side (lhs) while the right-hand side (rhs) holds terms with $\ddot{X}$ and $\ddot{\eta}$. So in order to simplify the coupled ODEs and also to perform some comparisons with other approximations, we suppose that the external force has a perturbational character, so it is sufficiently small. This fact entails small variations in time of both the velocity, $\dot{X}$, and the inverse of the width, η, so the approximation

$$\frac{d\log(\dot{X})}{dt}\sim 0 \quad\text{and}\quad \frac{d\log(\eta)}{dt}\sim 0 \tag{21}$$

can be made. Thus as a final result, in the limit of small external force, Eqs. (19) and (20) can be reduced to

$$\dot{\eta}=\frac{5(-1+\lambda\eta^2)^2\int_{-\infty}^{\infty}\text{sech}^2(z)F\,dz}{2\lambda\eta(-15-10\lambda\eta^2+\lambda^2\eta^4)\dot{X}}, \tag{22}$$

$$\begin{aligned}\dot{X}^2-1=&\frac{\lambda\eta^2}{1-\lambda\eta^2}-\frac{10(-1+\lambda\eta^2)}{\lambda\eta^3(-15-10\lambda\eta^2+\lambda^2\eta^4)}\\&\times\int_{-\infty}^{\infty}\{z\,\text{sech}^2(z)+\tanh(z)\\&+\lambda\eta^2[-z\,\text{sech}^2(z)+\tanh(z)]\}F\,dz,\end{aligned} \tag{23}$$

respectively. By carrying out the same procedure and approximations the corresponding ODEs for the quartic case ($n=4$) read

$$\dot{\eta}=\frac{3(1-\lambda\eta^2)^{3/2}\int_{-\infty}^{\infty}\text{sech}(z)F\,dz}{2\sqrt{2}\sqrt{\lambda}(\lambda^2\eta^4-6\lambda\eta^2-3)\dot{X}}, \tag{24}$$

$$\begin{aligned}\dot{X}^2-1=&\frac{\lambda\eta^2}{1-\lambda\eta^2}+\frac{3\sqrt{1-\lambda\eta^2}}{\sqrt{2}\sqrt{\lambda}\eta^2(\lambda^2\eta^4-6\lambda\eta^2-3)}\int_{-\infty}^{\infty}(z\,\text{sech}(z)\\&-\lambda\eta^2\{z\,\text{sech}(z)-2\arctan[\tanh(z/2)]\})F\,dz,\end{aligned} \tag{25}$$

with $z=\eta(x-X)$.

Equations (22)–(25) govern the dynamics of the soliton position X and the inverse soliton width η when a soliton moves in the presence of a small external force F. We note the important fact that these equations can also be obtained by using a more systematic method, namely adiabatic perturbation theory. In Appendix A we present, as an example, the adiabatic perturbation theory for the cubic case. As final result of this appendix we get Eqs. (22) and (23). A similar procedure can be followed for the quartic case to obtain the ODEs (24) and (25).

Notice that the external force F in Eqs. (22)–(25) can depend on time not only implicitly, via z, but also explicitly; e.g., in the case of thermal noise (Sec. V). Equations (22)–(25) are specialized for the particlelike properties of the soliton, so they cannot describe the trailing tail accompanying the soliton, which is generated when the soliton velocity is perturbed. In that case one must go to second-order perturbation theory [9,23].

Expressions similar to Eqs. (22)–(25) have been derived for the perturbed KdV equation [23–25]. However, due to the limitations of the KdV equation, those ODEs are only valid in the limit of soliton velocities very close to the sound velocity. Our equations are valid for a broader range of velocities since our theory has been developed in the framework of the IBq [6]. Notice that we have not taken into account the contribution of the force-induced phonons because the eigenvalue problem of the linearized stability equation of Bq type equations has not yet been completely solved [19]. However, in the case of high-energy solitons, whose velocity is high, the force-induced phonons may be negligible because those solitons are rather robust against perturbations [17,18]. On the other hand, in the case of low-energy solitons, whose velocity is very close to the sound velocity, the theory developed for the forced KdV equation, in which phonon contributions have been taken into account [24,25], can be used to study the soliton dynamics. So, in fact, those equations above, (22)–(25), are complementary to the equations already existent for the forced KdV equation. We show below that after some approximations our equations coincide with the equations developed for the forced KdV equation without the phonon contribution.

IV. LIMITING CASES: IPBQ EQUATION AND KDV EQUATION

In this section we investigate some limiting cases, namely the limit of small η together with the limit of soliton velocities very close to the sound velocity. Since the IPBq equation, (1), is a limiting case of the IBq equation, (2), we can expect that the corresponding ODEs for the collective coordinates for the forced IPBq,

$$\partial_t^2 y(x,t) - \partial_x^2 y(x,t) - \lambda \partial_x^4 y(x,t) - \partial_x(f(\partial_x y(x,t))) = F, \tag{26}$$

may also be an approximation of the ODEs (22)–(25). In fact, IPBq solitons and IBq solitons agree with each other when both are broad. In the case of narrower solitons the IBq equation is more accurate than the IPBq equation [6]. In other words, IBq solitons tend to be identical to IPBq solitons when η tends to be small. So in the approximation of small η, i.e., approximation to the second order Taylor expansion about $\eta=0$, Eqs. (22) and (23), the cubic case, become

$$\dot{\eta} = -\frac{1}{6\lambda \dot{X} \eta}\int_{-\infty}^{\infty} \operatorname{sech}^2(z) F dz, \tag{27}$$

$$\dot{X}^2 - 1 = \lambda \eta^2 - \frac{2}{3\lambda \eta^3}\int_{-\infty}^{\infty} [z \operatorname{sech}^2(z) + \tanh(z)] F dz, \tag{28}$$

and Eqs. (24) and (25), the quartic case, become

$$\dot{\eta} = -\frac{1}{2\sqrt{2}\sqrt{\lambda}\dot{X}}\int_{-\infty}^{\infty} \operatorname{sech}(z) F dz, \tag{29}$$

$$\dot{X}^2 - 1 = \lambda \eta^2 - \frac{1}{\sqrt{2}\sqrt{\lambda}\eta^2}\int_{-\infty}^{\infty} [z \operatorname{sech}(z)] F dz. \tag{30}$$

These equations, (27)–(30), can be recovered directly from Eq. (26) by using either the collective coordinate approach, similar to that developed in Sec. II, or adiabatic perturbation theory, similar to that developed in Appendix A.

If we go one step further and consider soliton velocities very close to the sound velocity ($\dot{X} \gtrsim 1$), one can show easily that in the cubic case Eqs. (27) and (28) become

$$\dot{\eta} = -\frac{1}{6\lambda \eta}\int_{-\infty}^{\infty} \operatorname{sech}^2(z) F dz, \tag{31}$$

$$\dot{s} = \frac{\lambda}{2}\eta^2 - \frac{1}{3\lambda \eta^3}\int_{-\infty}^{\infty} [z \operatorname{sech}^2(z) + \tanh(z)] F dz, \tag{32}$$

where $\dot{s} = \dot{X} - 1$ is the soliton velocity in the sound-velocity moving frame. Equations (31) and (32) are very similar to equations developed for the perturbed KdV equation [23–25]. Notice that the KdV equation is a completely integrable system whose eigenvalue problem is already solved [6,24,25]. Therefore in this framework the force-induced phonons can be taken into account. In fact, those force-induced phonons contribute with an extra term, namely $\tanh^2(z)$ [23–25], in the integrand of Eq. (32), so Eqs. (31) and (32) read

$$\dot{\eta} = -\frac{1}{6\lambda \eta}\int_{-\infty}^{\infty} \operatorname{sech}^2(z) F dz,$$

$$\dot{s} = \frac{\lambda}{2}\eta^2 - \frac{1}{3\lambda \eta^3}\int_{-\infty}^{\infty} [z \operatorname{sech}^2(z) + \tanh(z) + \tanh^2(z)] F dz. \tag{33}$$

Equations (33) can be derived by using either IST [25] or perturbation theory [24].

V. NOISY AND DAMPED IBQ EQUATION

As an example we consider the case of an IBq soliton in the presence of hydrodynamical damping and thermal noise, namely

$$F = \nu \partial_x^2 y(x,t) + D \partial_x \xi(x,t), \tag{34}$$

where the noise term $\xi(x,t)$ is delta-correlated, i.e.,

$$\langle \xi(x,t)\xi(x',t')\rangle = \delta(x-x')\delta(t-t'), \tag{35}$$

and

$$\langle \xi(x,t)\rangle = 0. \tag{36}$$

In Eqs. (35) and (36) $\langle \cdot \rangle$ means average. In order to satisfy the fluctuation-dissipation theorem the diffusion constant is defined as [17]

$$D = 2\nu T, \tag{37}$$

where T is the temperature.

Though here we are dealing with a continuum problem, namely a noisy IBq equation, it is worth mentioning that the discrete problem of diffusion of lattice solitons in monatomic chains can be approached by Eqs. (3) and (34). To the best of our knowledge this discrete diffusion problem has been analytically addressed only in the framework of the noisy KdV equation [17,18].

By substituting Eq. (34) in Eqs. (19) and (20) we get

$$\dot{\eta} = -\frac{\nu \eta^3(\lambda \eta^2 - 1)}{\lambda^2 \eta^4 - 10\lambda \eta^2 - 15} + \frac{5\sqrt{D}(-1+\lambda \eta^2)^2 \int_{-\infty}^{\infty} \mathrm{sech}^2(z)\partial_x \xi(x,t)dz}{2\lambda \eta(-15 - 10\lambda \eta^2 + \lambda^2 \eta^4)\dot{X}}, \quad (38)$$

$$\dot{X}^2 - 1 = \frac{\lambda \eta^2}{1-\lambda \eta^2} - 2\nu^2\eta^2 \frac{[30\pi^2 - 2(\pi^2-15)\lambda \eta^2 + \pi^2\lambda^2\eta^4]}{3(15 + 10\lambda \eta^2 - \lambda^2\eta^4)^2} - \frac{10\sqrt{D}(-1+\lambda \eta^2)}{\lambda \eta^3(-15 - 10\lambda \eta^2 + \lambda^2\eta^4)} \int_{-\infty}^{\infty} \{z\, \mathrm{sech}^2(z) + \tanh(z) + \lambda \eta^2[-z\, \mathrm{sech}^2(z) + \tanh(z)]\}\partial_x \xi(x,t)dz. \quad (39)$$

From Eqs. (38) and (39) it is straightforward to obtain

$$\langle \dot{\eta} \rangle = -\frac{\nu \langle \eta \rangle^3(\lambda \langle \eta \rangle^2 - 1)}{\lambda^2\langle \eta \rangle^4 - 10\lambda \langle \eta \rangle^2 - 15}, \quad (40)$$

$$\mathrm{Var}\left[\int dt(\dot{X}^2 - 1)\right] = D\int dt\left(\frac{10(\lambda\langle \eta \rangle^2 - 1)[30 + \pi^2 + \lambda\langle \eta \rangle^2(30 - 2\pi^2 + \pi^2\lambda\langle \eta \rangle^2)]}{9\lambda^2\langle \eta \rangle^3(15 + 10\langle \eta \rangle^2 - \langle \eta \rangle^4)^2}\right), \quad (41)$$

where $\mathrm{Var}(x) = \langle (x - \langle x \rangle)^2 \rangle$ is the variance function. If we define the relative velocity $\dot{s} = \dot{X} - 1$ we can use the approach $\dot{X}^2 - 1 \simeq 2\dot{s}$ if $\dot{s} \leq 0.2$, i.e., an error of about 10%. In that case

$$\mathrm{Var}(s) \simeq \frac{1}{4}\mathrm{Var}\left[\int dt(\dot{X}^2 - 1)\right], \quad (42)$$

where s is the position of the soliton in a frame moving with the sound velocity.

A. Simulations

In order to compare with our analytical results we have performed numerical simulations of the IBq equation in the presence of noise and damping. We note that Eq. (3) is written in terms of the field $y(x,t)$ which has the shape of a kink for the one-soliton solutions (6). Notice that the one-soliton solutions here move with supersonic velocities. So, for long time scales it would be necessary to use large numerical systems, which is too time-consuming. Moreover, round-off errors can be a serious drawback for the simulation of large systems, since in our algorithm a matrix inversion is needed (see Appendix B for details). In order to avoid these problems mentioned above the use of periodic boundary conditions is desirable, so a small numerical system can be achieved. On the other hand, a pulse-shape form of the soliton is suitable for simulating the system, since this shape vanishes at infinity. The pulse shape can be obtained by derivating the kink form with respect to x, i.e., $u(x,t) = \partial_x y(x,t)$. So, in this case the equation of motion for the function $u(x,t)$ takes the form

$$\partial_t^2 u(x,t) - \partial_x^2 u(x,t) - \lambda \partial_x^2 \partial_t^2 u(x,t) - \partial_x^2(f(u(x,t))) = \partial_x F, \quad (43)$$

where $f(u)$ is given in Eq. (4), and the external force F given in Eq. (34). Notice that the field $u(x,t)$ can be associated, for instance, with the *relative* displacement between adjacent particles in a classical chain of atoms. In order to discretize Eq. (43) we have followed the procedure proposed in Ref. [9] with periodic boundary conditions. See Appendix B for details.

We remark that the length L of the numerical system has been chosen to be $L \gg \eta^{-1}$, where η^{-1} is the width of the soliton. Typical values are $L=500$ and $\eta^{-1} \sim 1$. Other parameters are the dispersion parameter $\lambda=1$, the spatial mesh size $\Delta x = 0.0625$, the time step of the integrator $\Delta t = 10^{-2}$, and the number of realizations is 100.

B. Soliton position

In order to study our noisy system numerically we perform statistics over an ensemble of a finite number of realizations. The numerical detection of the position of a soliton in the presence of noise in each realization is an extremely difficult task because the IBq soliton is a nontopological excitation, whose position is very robust against perturbations. Since the soliton is nontopological its amplitude reduces with time when a damping term is present. So for larger scales of time the soliton profile is strongly masked by the thermal noise. On the other hand the soliton position is not so much affected by the noise term, which results in small values of the variance of the soliton position. So, the standard deviation of the soliton position is usually much smaller than the soliton width, making very difficult its numerical estimate. Notice that since the pulse shape is strongly masked by the noise, important features like the soliton amplitude or the width cannot be directly determined in each realization. We note, however, that the ensemble average yields a well-defined soliton shape, where the averaged amplitude, averaged soliton position, $\bar{x}_S$, or averaged soliton width, $\bar{w}_S$, can be estimated without difficulty.

In order to estimate numerically the soliton position in each realization we have used a filtering process as follows.

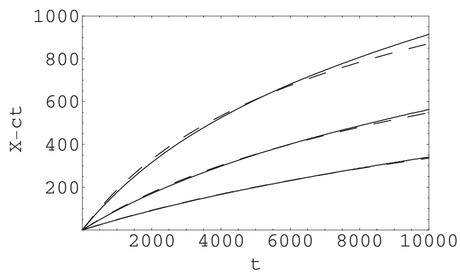

FIG. 1. Soliton position in a frame moving with the sound velocity ($c=1$) vs time for three different initial velocities, namely $v(0)=1.05$, 1.1, and 1.2. Solid lines: simulations and dashed lines: theory.

For a given time t we have multiplied the noisy soliton shape $u(x,t)$ by a Gaussian window function,

$$G(x-x_G,t)=\exp\left[-\frac{1}{2}\left(\frac{x-x_G}{w_G}\right)^2\right]. \tag{44}$$

In order to reduce the undesirable effect of the noisy wings of the soliton we have set $w_G=\bar{w}_S/4$. So in order to determine the position of the soliton we are concerned only with the noisy central part of the soliton shape, i.e., $\bar{x}_S-\bar{w}_S/2 \leq x \leq \bar{x}_S+\bar{w}_S/2$. Afterwards, we calculate the center of mass $\bar{x}_G$ of the function $u_G(x-x_G,t)=u(x,t)G(x-x_G,t)$, i.e.,

$$\bar{x}_G=\int_{-\infty}^{\infty} u_G(x-x_G,t)x\,dx. \tag{45}$$

Notice that in Eq. (45) it would be desirable that the center x_G of the Gaussian window agrees with the center of the soliton; but, since we do not know exactly the soliton position we adopt an averaging procedure to calculate it. That is, we calculate the center-of-mass position $\bar{x}_G$ for different positions x_G of the Gaussian window function in a region close to the average soliton position, i.e., $\bar{x}_S-\bar{w}_S/2 \leq x_G \leq \bar{x}_S+\bar{w}_S/2$. Afterwards we average these values, so

$$\bar{x}=\frac{1}{\bar{w}_S}\int_{\bar{x}_s-\bar{w}_S/2}^{\bar{x}_s+\bar{w}_S/2} \bar{x}_G\,dx_G, \tag{46}$$

where $\bar{x}$ is defined as the soliton position.

VI. RESULTS

In Figs. 1 and 2 we present a comparison between theory and simulations for the soliton position and its variance, respectively. In particular in Fig. 1 we observe a very good agreement of the soliton positions for different soliton velocities. Small discrepancies between simulations and theory are observed for high velocities, namely $\dot{X}(0)=1.1$ and 1.2. We note that some of the relative soliton velocities $\dot{X}-1$ here are about ten times larger that those considered in Ref. [18] in the frame of the KdV equation.

In Fig. 2 we show the variance of the soliton position for different values of the temperature and the initial soliton ve-

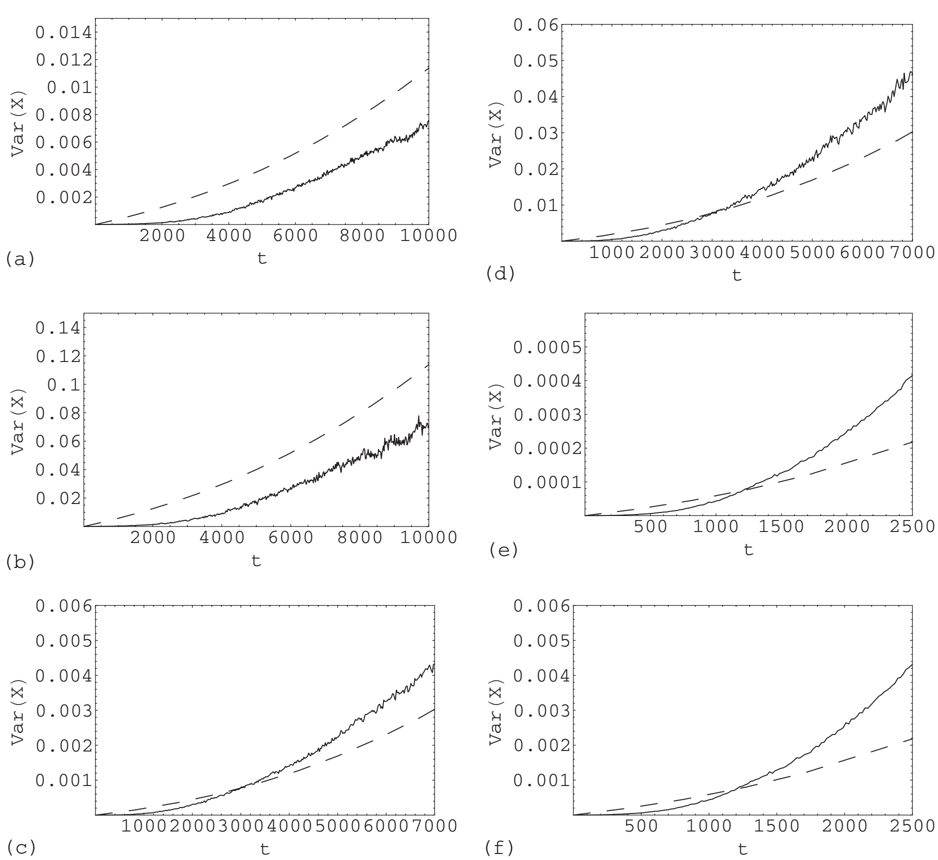

FIG. 2. Variance of the position vs time for two different temperatures, namely $T=10^{-5}$ [panels (a), (c), and (e)] and 10^{-4} [panels (b), (d), and (f)], and three different initial velocities, namely $v(0)=1.05$ [panels (a) and (b)], 1.1 [panels (c) and (d)], and 1.2 [panels (e) and (f)]. Solid lines: simulations and dashed lines: theory.

locity. Our analytical prediction, namely the solution of Eqs. (40)–(42), is also plotted for comparison. We observe a superdiffusive behavior (nonlinear growth in time) of the variance which scales with the temperature. We also observe that our theory predicts the qualitative behavior of the variance better for low initial soliton velocities ($\dot{X} \leq 1.05$) than for the higher ones ($\dot{X} > 1.05$). In fact, the approximation performed in Eq. (41) is met better in the low-velocity regime.

It is known from theory of monatomic chains that thermal noise affects the dynamics of solitons in two different forms. In the first form the soliton is "kicked" by the stochastic fluctuations randomly changing its position but not its velocity. In this case the diffusion is normal, i.e., the variance grows linearly in time. This dynamics has been observed for very low-velocity solitons [$\dot{X}(0) < 1.005$] [17], which have a broad width. In the second form not only the position of the soliton is affected but also its velocity, which in turn affects the soliton position, too. The stochastic changes in the velocity are mostly due to the stochastic distortions of the soliton width. This dynamics is observed for high-velocity solitons [$\dot{X}(0) > 1.005$] [17,18,20] whose width is narrow. Notice that the soliton velocity $\dot{X}$ and the soliton width η^{-1} are coupled in a nontrivial form as we show, for instance, in Eq. (39). The stochastic changes of the velocity contributes to the variance of the position, making it grow nonlinearly, i.e., superdiffusion is observed [17,18,20].

The dynamics described above can be interpreted as follows. The velocity of very broad solitons [$\dot{X}(0) < 1.005$] tends to "experience" the stochastic fluctuations in average, which is zero (only normal diffusion is observed) [17]. On the other hand, the velocity of narrow solitons [$\dot{X}(0) > 1.005$] tends to directly "experience" the stochastic fluctuations, even if the fluctuations are very small with respect to the amplitude of the soliton (superdiffusion is observed). Notice that since the theory developed here is more appropriate for high-velocity solitons, we are concerned mostly with the superdiffusive behavior of the solitons.

If we compare in Fig. 2 the left-hand-side panels with the right-hand-side panels, we observe that for the same velocity the values of the variance scale with the temperature, as mentioned above. On the other hand, if we compare the behavior of the variance for the same temperature but different velocities [see Figs. 2(a), 2(c), and 2(e) or Figs. 2(b), 2(d), and 2(f)], the superdiffusive behavior is more pronounced, i.e., the rise of the variance in time is faster for high velocities [Figs. 2(e) and 2(f)] than for low velocities [Figs. 2(a) and 2(b)]. Notice, however, that the overall values of the variance are smaller for high velocities [Figs. 2(e) and 2(f)] than for the lower ones [Figs. 2(a) and 2(b)]. This is because high-velocity solitons are more energetic and robust against perturbations than the low-velocity ones.

The overall behavior of the soliton diffusion shown above can be summarized by saying that the absolute values of the variance depend on the temperature, while the superdiffusive behavior, i.e., the rapidity how the variance rises, depends mostly on the soliton velocity.

Since the scale of the variance is small and also depends strongly on the soliton velocity, small discrepancies of the velocity between theory and simulations do not affect so much the prediction of the soliton position (see Fig. 1), but can lead to considerable discrepancies in the variance for high soliton velocities [see Figs. 2(e) and 2(f)]. On the other hand, the detection of the soliton position [see Eqs. (45) and (46)] depends on the soliton shape which in turn is masked by the thermal noise. So, even for robust solitons (solitons with high initial soliton velocity, i.e., high amplitude) small errors in the detection of the soliton position can induce observable errors of the variance due to the small scales we are dealing with.

VII. SUMMARY

We studied analytically the case of Boussinesq solitons in the presence of an external force. In order to do so we considered both the forced improved Boussinesq equation (forced IBq) and the forced ill-posed Boussinesq equation (forced IPBq). We used a collective coordinate approach to get equations of motion for the relevant variables of the soliton, namely position, X, and inverse width, η. We showed the explicit calculations for the forced IBq with cubic anharmonicity. We got coupled ODEs for the collective variables, Eqs. (19) and (20). Using physical considerations we reduced those ODEs to Eqs. (22)–(25) when the external force was considered as a perturbation. Moreover, we showed that those equations, (22)–(25), can also be derived by using a systematic method, namely adiabatic perturbation theory (Appendix A).

In the limit of small η those equations lead to similar ODEs, (27)–(30), which are associated with the forced IPBq equation. They can be derived directly from the forced IPBq by using either a collective variable approach or perturbation theory.

In the limit of soliton velocities very close to the sound velocity, Eqs. (27)–(30) lead to Eqs. (31) and (32) which are expressions already known in the context of the perturbed KdV equation. Those equations, (31) and (32), do not take into account the phonon contributions which for the KdV equation can be calculated. An extra term must be added in order to take into account the force-induced phonons; compare Eqs. (31) and (32) with Eqs. (33). Notice that the high-velocity solitons are rather robust against perturbations, therefore for them the effect of the force-induced phonons may be negligible. In that case Eqs. (22)–(25) govern the soliton dynamics in the presence of a perturbational external force. In the case of low-velocity solitons, where force-induced phonons cannot be negligible, Eqs. (33) describe the soliton dynamics better than Eqs. (31) and (32). Under those considerations one may consider Eqs. (22)–(25) as complementary to Eqs. (33).

Moreover, we consider an external force which consists of noise and damping. Similar problems have been considered for discrete systems, namely chains of atoms [17,18,20]. Indeed, the case of soliton diffusion in monatomic chains was addressed analytically in the frame of the KdV equation [18] with the help of Eqs. (33).

Here, we considered initial soliton velocities larger than those the KdV theory can correctly deal with. We showed that the numerical simulations can be described by the theory developed here. We also mentioned that the variance of the soliton position is superdiffusive and scales with the temperature. A similar behavior has been observed for lattice solitons in chains of atoms.

Finally, we note that the behavior of solitons of the IBq equation, which is a continuous system, and the behavior of lattice solitons in chains of atoms is rather similar and can be described by the same set of equations, derived here in the frame of the collective coordinate approach.

APPENDIX A: ADIABATIC PERTURBATION EXPANSION

In this appendix we apply an adiabatic perturbation theory to the perturbed IBq,

$$\partial_t^2 y - \partial_x^2 y - \lambda \partial_t^2 \partial_x^2 y - \partial_x^2 (f(\partial_x y)) = \epsilon F. \tag{A1}$$

Here, $f(u)=\frac{dV(u)}{du}-u$ is a nonlinear force where $u=\partial_x y$. The rhs of Eq. (A1) represents the action of the external force. The parameter ϵ is introduced to keep track of the different orders of the perturbation theory. Notice that in the case $\epsilon=1$ we get our original forced IBq, Eq. (3). In our derivation we will follow partially the procedure which was proposed in Ref. [24] for the perturbed Korteweg-de-Vries equation. The calculations that we present here are only for the case of a cubic anharmonicity, however, the same procedure can be performed easily for the quartic anharmonicity.

We define the center of mass position of the soliton as

$$X = x_1(\bar{t}) + x_0(T), \tag{A2}$$

where

$$\bar{t} = t, \quad T = \epsilon t, \tag{A3}$$

$$x_1(\bar{t}) = \int_0^{\bar{t}} c(\epsilon \bar{t}') d\bar{t}' \quad \text{and} \quad c(T) = \frac{1}{\sqrt{1-\lambda \eta^2(T)}}. \tag{A4}$$

Here, x_0 and η, which is the inverse of the width soliton, depend on the "slow" time variable T, and x_1 depends on the "fast" time variable $\bar{t}$. Notice the fact that

$$\partial_t = \partial_{\bar{t}} + \epsilon \partial_T. \tag{A5}$$

We seek an asymptotic solution in the form

$$y = y_0 + \epsilon y_1 + \cdots. \tag{A6}$$

So inserting Eqs. (A5) and (A6) into Eq. (A1) and collecting powers of ϵ we get

ϵ^0:

$$\partial_{\bar{t}}^2 y_0 - \partial_x^2 y_0 - \lambda \partial_x^2 \partial_{\bar{t}}^2 y_0 - \partial_x (f(\partial_x y_0)) = 0, \tag{A7}$$

ϵ^1:

$$\partial_{\bar{t}}^2 y_1 - \partial_x^2 y_1 - \lambda \partial_x^2 \partial_{\bar{t}}^2 y_1 - \partial_x (f'(\partial_x y_0)\partial_x y_1) = G, \tag{A8}$$

where

$$G = F - 2\partial_{\bar{t}} \partial_T y_0 + 2\lambda \partial_x^2 \partial_{\bar{t}} \partial_T y_0. \tag{A9}$$

The one-soliton solution of Eq. (A7), in the case of $n=3$, takes the form

$$y_0 = \frac{3\lambda \eta}{1-\lambda \eta^2} \tanh\left(\frac{\eta}{2}(x-X)\right), \tag{A10}$$

where X is defined by Eq. (A3).

Here, we use Green's theorem [26],

$$v \mathcal{L} y_1 - y_1 \tilde{\mathcal{L}} v = \text{divergence}, \tag{A11}$$

where v is any function and the linear operator

$$\mathcal{L} = \partial_{\bar{t}}^2 - \partial_x^2 - \lambda \partial_x^2 \partial_{\bar{t}}^2 - \partial_x (f'(\partial_x y_0)\partial_x), \tag{A12}$$

and $\tilde{\mathcal{L}}=\mathcal{L}$ is its adjoint.

If we demand that

$$\tilde{\mathcal{L}} v = 0 \tag{A13}$$

and we integrate Eq. (A11) over space, we can derive a compatibility condition of the form

$$\int_{-\infty}^{\infty} dx v \mathcal{L} y_1 = \int_{-\infty}^{\infty} dx v G = 0. \tag{A14}$$

Equation (A13) has two linear independent solutions, namely

$$v_1 = \partial_x y_0, \tag{A15}$$

$$v_2 = \partial_\eta y_0. \tag{A16}$$

Inserting the solutions (A15) and (A16) into Eq. (A14) we get

$$\frac{dx_1(\bar{t})}{d\bar{t}} \frac{d\eta(T)}{dT} = \frac{5[-1+\lambda \eta^2(T)]^2 \int_{-\infty}^{\infty} \text{sech}^2(z) F dz}{2\lambda \eta(T)[-15 - 10\lambda \eta^2(T) + \lambda^2 \eta^4(T)]} \tag{A17}$$

and

$$\frac{dx_1(\bar{t})}{d\bar{t}} \frac{dx_0(T)}{dT} = -\frac{5[-1+\lambda \eta^2(T)]}{\lambda \eta^3(T)[-15 - 10\lambda \eta^2(T) + \lambda^2 \eta^4(T)]} \times \int_{-\infty}^{\infty} \{z\, \text{sech}^2(z) + \tanh(z) + \lambda \eta^2(T)[-z\, \text{sech}^2(z) + \tanh(z)]\} F dz, \tag{A18}$$

respectively. Here, $z=\frac{\eta(T)}{2}[x-x_1(\bar{t})-x_0(T)]$. If we recover our original time variable t, from Eqs. (A3) and (A5) one can show that

$$\dot{X} = \frac{dx_1(\bar{t})}{d\bar{t}} + \epsilon \frac{dx_0(T)}{dT}. \tag{A19}$$

Therefore up to first order we get

$$\dot{X}\dot{\eta} = \epsilon \frac{dx_1(\bar{t})}{d\bar{t}} \frac{d\eta(T)}{dT}, \tag{A20}$$

where $\dot{} \equiv \frac{d}{dt}$. Inserting Eq. (A17) into Eq. (A20) we get

$$\dot{\eta} = \epsilon \frac{5[-1+\lambda\eta^2(T)]^2 \int_{-\infty}^{\infty} \text{sech}^2(z) F dz}{2\lambda\eta(T)[-15-10\lambda\eta^2(T)+\lambda^2\eta^4(T)]\dot{X}}. \quad \text{(A21)}$$

On the other hand, we see that

$$\dot{X}^2 = \left(\frac{dx_1(\bar{t})}{d\bar{t}}\right)^2 + 2\epsilon \frac{dx_1(\bar{t})}{d\bar{t}} \frac{dx_0(T)}{dT}. \quad \text{(A22)}$$

So by inserting Eq. (A18) into Eq. (A22) and taking into account the definition (A4) it is straightforward to get

$$\dot{X}^2 - 1 = \frac{\lambda\eta^2(T)}{1-\lambda\eta^2(T)} - \epsilon \frac{10[-1+\lambda\eta^2(T)]}{\lambda\eta^3(T)[-15-10\lambda\eta^2(T)+\lambda^2\eta^4(T)]} \times \int_{-\infty}^{\infty} \{z\,\text{sech}^2(z) + \tanh(z) + \lambda\eta^2(T)[-z\,\text{sech}^2(z) + \tanh(z)]\} F dz. \quad \text{(A23)}$$

In order to interpret our results physically we must set ϵ to unity ($\epsilon=1$) and assume that the external force, F, is sufficiently small. In that case Eqs. (A21) and (A23) are equal to Eqs. (22) and (23), respectively.

APPENDIX B: DISCRETIZATION OF THE IBQ EQUATION

The IBq equations reads

$$\partial_t^2 u(x,t) - \partial_x^2 u(x,t) - \partial_t^2\partial_x^2 u(x,t) - \partial_x^2(f(u(x,t))) = K(x,t), \quad \text{(B1)}$$

where $K(x,t)$ are external forces and/or dissipation. By defining the variable $v(x,t)=\partial_t u(x,t)$ Eq. (B1) can be reduced to two partial differential equations of first order in time, namely

$$\partial_t v(x,t) = \partial_x^2 u(x,t) + \partial_x^2 \partial_t v(x,t) + \partial_x^2(f(u(x,t))) + K(x,t),$$

$$\partial_t u(x,t) = v(x,t). \quad \text{(B2)}$$

By using finite-difference discretization in the space-domain Eqs. (B2) take the form

$$\dot{v}_i(t) = \frac{u_{i+1}(t)-2u_i(t)+u_{i-1}(t)}{\Delta x^2} + \frac{\dot{v}_{i+1}(t)-2\dot{v}_i(t)+\dot{v}_{i-1}(t)}{\Delta x^2} + \frac{f(u_{i+1}(t))-2f(u_i(t))+f(u_{i-1}(t))}{\Delta x^2} + K_i(t),$$

$$\dot{u}_i(t) = v_i(t), \quad \text{(B3)}$$

where $\dot{} \equiv \frac{d}{dt}$, $u_i(t)=u(x_i,t)$, $v_i(t)=v(x_i,t)$, $f(u_i(t))=u^n(x_i,t)$, and $K_i(t)=K(x_i,t)$ with $n=2,3$. $x_i=i\Delta x$ where Δx is the mesh size of the space variable and $i=1,2,\ldots,N$. The length of the system $L=N\Delta x$. In the numerical integration process we use periodic boundary conditions, namely $u_0(t)=u_N(t)$ and $u_{N+1}(t)=u_1(t)$. The same boundaries are used for the variables $v_i(t)$ and $F_i(t)$ including the noise term. If we rewrite Eqs. (B3) so

$$-\dot{v}_{i+1}(t) + (\Delta x^2+2)\dot{v}_i(t) - \dot{v}_{i-1}(t) = u_{i+1}(t) - 2u_i(t) + u_{i-1}(t) + f(u_{i+1}(t)) - 2f(u_i(t)) + f(u_{i-1}(t)) + \Delta x^2 K_i(t), \quad \text{(B4)}$$

$$\dot{u}_i(t) = v_i(t), \quad \text{(B5)}$$

they can be regarded as a vectorial equations, so

$$\hat{\mathbf{A}}\dot{\mathbf{v}} = \mathbf{G}, \quad \dot{\mathbf{u}} = \mathbf{v}, \quad \text{(B6)}$$

where $\dot{u}_i$ and $\dot{v}_i$ are elements of the vectors $\dot{\mathbf{u}}$ and $\dot{\mathbf{v}}$, respectively. The elements G_i of the vector $\mathbf{G}$ are the rhs of Eq. (B4) and the square matrix

$$\hat{\mathbf{A}} = \begin{pmatrix} \Delta & -1 & 0 & 0 & \cdots & \cdots & 0 & 0 & -1 \\ -1 & \Delta & -1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & \cdots & 0 & -1 & \Delta & -1 & 0 & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & \cdots & 0 & 0 & -1 & \Delta & -1 \\ -1 & 0 & 0 & 0 & \cdots & \cdots & 0 & -1 & \Delta \end{pmatrix}_{N\times N}$$

with $\Delta=\Delta x^2+2$. Notice that this tridiagonal matrix is cyclic because we use periodic boundary conditions [28]. From Eq. (B6) we can derive

$$\dot{\mathbf{v}} = \hat{\mathbf{A}}^{-1}\mathbf{G}, \quad \dot{\mathbf{u}} = \mathbf{v}, \quad \text{(B7)}$$

therefore at this stage we can use a classical integrator as, for example, the Heun algorithm [27] in order to perform the numerical integration in time.

We note also that for the discretization of the noise term we use the definition [18]

$$\partial_x^2 \xi(x,t) \to \frac{\xi_{i+1}(t) - 2\xi_i(t) + \xi_{i-1}(t)}{\Delta x^{5/2}}, \quad \text{(B8)}$$

where $\xi_i(t)$ is the Gaussian white noise term at the i mesh point, which is generated by a random number generator with normal distribution.

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